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Forced Time-Dependent Harmonic Oscillator in a Static Magnetic Field: Exact Quantum and Classical Solutions

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Exact wave functions of the forced time-dependent two-dimensional harmonic oscillator in a static magnetic field are derived by unitary transformation. The geometrical phase induced by the driving force is the phase of the de Broglie wave associated with the particle moving according to the classical equation. Extending the idea of the Heisenberg correspondence principle to the time-dependent system, the exact classical solution is obtained from quantum matrix elements.

KEY WORDS: harmonic oscillator; magnetic field; exact wave function; exact classical solution.

1. INTRODUCTION

The harmonic oscillator is a central topic both in classical mechanics and quantum mechanics. A great many investigations have been done about the exact wave functions of the time dependent harmonic oscillator and its modifications (Brown, 1991; Feng and Wang, 1995; Ferreira *et al.*, 2002; Gweon and Choi, 2003; Husimi, 1953; Kiss *et al.*, 1994; Lai *et al.*, 1996; Lewis, 1967; Lewis and Riesenfeld, 1969; Liang and Wu, 2003; Lo, 1993a,b; Mizrahi, 1989; Pedrosa, 1997; Wang *et al.*, 2000; Yu *et al.*, 1998). Recently, Ferreira *et al.* (2002) obtained the exact wave functions of a time-dependent harmonic oscillator in a static magnetic field. In this article, we deal with the problem that there is a time-dependent external force $F(t)$. Besides the exact wave functions, we derive the exact classical solution by extending the Heisenberg correspondence principle to the time-dependent systems (Greenberg and Klein, 1995; Huang, 1986; Morehead, 1996).

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For a time-independent system, the Heisenberg correspondence principle says that the matrix elements of a physical quantity gives the coefficients of the Fourier expansions of the physical quantity in the classical limit (Greenberg and Klein, 1995; Huang, 1986; Morehead, 1996). In another word, the quantum matrix element can give the classical solution. It is interesting to note that such a phenomenon can be used for time-dependent system (Liang and Wu, 2003). When both the exact wave functions and the exact classical solution are obtained, the relationship between the quantum and classical phases appears automatically.

This article is organized as follows. The next section gives the derivation of the exact wave functions. The quantum matrix element and the classical limit are in the third section and the final section is the conclusion.

2. THE EXACT WAVE FUNCTIONS

From the Hamiltonian of the time-dependent harmonic oscillator in a static magnetic field (Ferreira *et al.*, 2002) we easily get the Hamiltonian for the forced system

$$
\begin{aligned}
H &= H_0(t) - F(t)x \\
H_0(t) &= \frac{p_x^2 + p_y^2}{2M(t)} + \frac{1}{2}\varpi_c L_z + \frac{1}{2}M(t)\Omega(t)^2(x^2 + y^2)
\end{aligned} \tag{1}
$$

where $F(t)$ is the external force, p_x and p_y are the momentum operators, $L_z = xp_x - yp_x$ is the angular momentum operator in the axial z direction, $\varpi_c = eB_0(t)/M(t)$ is the cyclotron frequency of oscillation. The general frequency $\Omega(t)$ takes the form

$$
\Omega^2(t) = \frac{1}{4}\varpi_c^2 + \omega^2(t) \tag{2}
$$

For the undriven system $H_0(t)$, the exact wave functions are (Ferreira *et al.*, 2002)

$$
\begin{aligned}
\psi_{nm}^0(x, y, t) &= \exp[i\alpha_{nm}(t)]\phi_{nm}(x, y, t) \\
\phi_{nm}(x, y, t) &= \frac{1}{\rho\sqrt{\hbar}}\varphi_{nm}(\eta, \xi)\exp\left[i\frac{M\dot{\rho}}{2\hbar\rho}(x^2 + y^2)\right]
\end{aligned} \tag{3}
$$

where $n = 0, 1, 2, \ldots$ and $m = 0, \pm 1, \pm 2, \ldots$. Here, the variables η and ξ are defined as $\eta = x/(\rho\sqrt{\hbar})$, $\xi = y/(\rho\sqrt{\hbar})$. The auxiliary equation for the function $\rho(t)$ is

$$
\ddot{\rho} + \frac{\dot{M}\dot{\rho}}{M} + \Omega^2\rho = \frac{1}{M^2\rho^3} \tag{4}
$$

The function $\varphi_{nm}(\eta, \xi)$ satisfies

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial\eta^2}+\frac{\partial^2}{\partial\xi^2}\right)+\frac{1}{2}(\eta^2+\xi^2)+\rho^2\frac{eB_0L_z}{2\hbar}\right]\varphi_{nm}(\eta,\xi)$$
$$=\left(N+1+\rho^2\frac{eB_0}{2}m\right)\varphi_{nm}(\eta,\xi) \tag{5}$$

where $N = 2n + |m|$. The phase factor in (3) has the form

$$\alpha_{nm}(t) = \alpha_{nm}^{(1)}(t) + \alpha_{nm}^{(2)}(t)$$
$$\alpha_{nm}^{(1)}(t) = -m\int_0^t \frac{\varpi_c(\tau)}{2}d\tau \tag{6}$$
$$\alpha_{nm}^{(2)}(t) = -(N+1)\int_0^t \frac{\tau}{M(\tau)\rho^2(\tau)}$$

In cylindrical coordinates $\eta = R\cos\phi$, $\xi = R\sin\phi$, $\varphi_{nm}(\eta, \xi)$ can be further written as $\varphi_{nm}(\eta, \xi) = \varphi_{Nm}(R)e^{im\phi}/\sqrt{2\pi}$ and Eq. (5) becomes

$$\left[-\frac{1}{2}\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right)+\frac{m^2}{2R^2}+\frac{1}{2}R^2\right]\varphi_{Nm}(R) = (N+1)\varphi_{Nm}(R) \tag{7}$$

Clearly, (7) is the radial equation of a two-dimensional harmonic oscillator with unit mass, frequency and Planck constant.

If there exists an external force, we shall show that the exact wave functions are unitary transformations of (3). For the case $F = 0$, one can easily prove

$$\frac{dL_z}{dt} = \frac{\partial L_z}{\partial t} + \frac{1}{i\hbar}[L_z, H] = 0 \tag{8}$$

which means that there is another invariant, the angular momentum, besides the Ermakov invariant $I = I(x,t) + I(y,t)$ (Ferreira *et al.*, 2002). Really, the wave function (3) is the eigenfunction of L_z. So, $\phi_{nm}(x, y, t)$ is the common eigenfunction of the invariants L_z and I, since $\phi_{nm}(x, y, t)$ is the eigenfunction of the invariant I too (Ferreira *et al.*, 2002). The condition $[L_z, I] = 0$ for L_z and I to have the common eigenfunctions can be proved by straightforward calculations.

If the external field is applied to the system or $F \neq 0$, the following operator will be a new invariant

$$L'_z = (x - f_1(t))(p_y - g_2(t)) - (y - f_2(t))(p_x - g_1(t)) \tag{9}$$

It is not difficult to show that $dL'_z/dt = 0$, with $f_i(t), g_i(t), i = 1, 2$ satisfying the classical equations of motion

$$\dot{f}_1 = \frac{g_1}{M} - \frac{1}{2}\varpi_c f_2$$

$$\begin{aligned}\dot{g}_1 &= -M\Omega^2 f_1 + F(t) - \frac{1}{2}\varpi_c g_2 \\ \dot{f}_2 &= \frac{g_2}{M} + \frac{1}{2}\varpi_c f_1 \\ \dot{g}_2 &= -M\Omega^2 f_2 + \frac{1}{2}\varpi_c g_1\end{aligned} \tag{10}$$

which may be further cast into the equations for the displacements $f_i(t), i = 1, 2$

$$\begin{aligned}\ddot{f}_1 + \frac{\dot{M}}{M}\dot{f}_1 + \omega^2 f_1 &= -\varpi_c \dot{f}_2 + F(t) \\ \ddot{f}_2 + \frac{\dot{M}}{M}\dot{f}_2 + \omega^2 f_2 &= -\varpi_c \dot{f}_1\end{aligned} \tag{11}$$

The operator L'_z is connected with the angular momentum L_z through the unitary transformation

$$\begin{aligned}L'_z &= U^+ L_z U \\ U &= \exp\left[i\frac{f_1}{\hbar}p_x + i\frac{f_2}{\hbar}p_y\right]\exp\left[-i\frac{g_1}{\hbar}x - i\frac{g_2}{\hbar}y\right]\end{aligned} \tag{12}$$

where the following relations are used

$$\begin{aligned}U^+ x U &= x - f_1, \quad U^+ p_x U = p_x - g_1 \\ U^+ y U &= y - f_2, \quad U^+ p_y U = p_y - g_2\end{aligned} \tag{13}$$

Similarly, the invariant I transforms into $I' = U^+ I U$. After the unitary transformation, the common eigenfunction $\phi_{nm}(x, y, t)$ of the invariants L_z, and I becomes

$$U^+ \phi_{nm}(x, y, t) \tag{14}$$

which is the common eigenfunction of the new invariants L'_z and I'. Hence, the wave function for the forced system can be written in the form

$$\psi_{nm}(x, y, t) = \exp[i\beta(t)]\exp[i\alpha_{nm}(t)]U^+\phi_{nm}(x, y, t) \tag{15}$$

where $\beta(t)$ is a new phase factor caused by the external force. The new phase is calculated through (Lewis, 1967; Lewis and Riesenfeld, 1969; Mizrahi, 1989)

$$\dot{\alpha}_{nm}(t) + \dot{\beta}(t) = \frac{1}{\hbar}\left\langle \phi_{nm} | U \left(i\hbar\frac{\partial}{\partial t} - H\right) U^+ | \phi_{nm}\right\rangle \tag{16}$$

To calculate the phase, the relations (13) and the following averages are used

$$\begin{aligned}\langle\phi_{nm}|x|\phi_{nm}\rangle &= 0, \quad \langle\phi_{nm}|y|\phi_{nm}\rangle = 0 \\ \langle\phi_{nm}|p_x|\phi_{nm}\rangle &= 0, \quad \langle\phi_{nm}|p_y|\phi_{nm}\rangle = 0\end{aligned} \tag{17}$$

The phase is divided into two parts: the dynamical part $\beta_d(t)$ and the geometrical part $\beta_g(t)$

$$\beta_d(t) = \frac{1}{\hbar}\int_0^t \left\{\frac{g_1^2 + g_2^2}{2M} + \frac{1}{2}\varpi_c(f_1 g_2 - f_2 g_1) + \frac{1}{2}M\Omega^2\left(f_1^2 + f_2^2\right) - f_1 F\right\} d\tau$$

$$\beta_g(t) = \frac{1}{\hbar}\int_0^t \left\{f_1(\tau)\frac{dg_1(\tau)}{d\tau} + f_2(\tau)\frac{dg_2(\tau)}{d\tau}\right\} d\tau \tag{18}$$

The geometric part can be further written as

$$\beta_g(t) = -\frac{1}{\hbar}\int_0^t d\{f_1(\tau)g_1(\tau) + f_2(\tau)g_2(\tau)\}$$
$$+\frac{1}{\hbar}\int_0^t \{g_1(\tau)df_1(\tau) + g_2(\tau)df_2(\tau)\} \tag{19}$$

The first term is integrible and can be removed by redefining the phase of $\phi_{nm}(x, y, t)$

$$\phi_{nm}(x, y, t) \to e^{-i\frac{1}{\hbar}(f_1 g_1 + f_2 g_2)}\phi_{nm}(x, y, t) \tag{20}$$

The second term in (19) is in fact the phase of the de Brolie wave of the particle moving according to the classical equations (10) or (11).

3. MATRIX ELEMENTS AND THE CLASSICAL LIMIT

To simplify the expressions, now we use the Dirac notes. Following Liang and Wu (2003), define a wave packet

$$|\varphi(t)\rangle = \sum_{Nm} |\psi_{Nm}(t)\rangle \tag{21}$$

and a quantity

$$O_{Nm}(t) = \mathrm{Re}\langle\varphi(t)|O|\psi_{Nm}(t)\rangle = \mathrm{Re}\sum_{N'm'}\langle\psi_{N'm'}(t)|O|\psi_{Nm}(t)\rangle \tag{22}$$

where the quantum number n is replaced by N through $N = 2n + |m|$, O is the operator in Schröndinger picture for a physical variable, "Re" means the real part is taken. Using the Schröndinger equation

$$H|\psi_{Nm}(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi_{Nm}(t)\rangle, \quad H|\varphi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\varphi(t)\rangle \tag{23}$$

one can get the following evolution equation for $O_{Nm}(t)$

$$\frac{dO_{Nm}(t)}{dt} = \mathrm{Re}\langle\varphi(t)|\frac{1}{i\hbar}[O, H]\psi_{Nm}(t)\rangle \tag{24}$$

Setting $O = x, y, p_x$, and p_y, we get equations similar to the classical Eq. (10) for $x_{Nm}(t)$, $y_{Nm}(t)$, $(p_x)_{Nm}(t)$, and $(p_y)_{Nm}(t)$. Below we demonstrate that in the classical limit, $O_{Nm}(t)$ agrees with the classical solution. Using $O^0_{Nm}(t)$ to express the quantity for the undriven system, the following relations can be derived

$$\begin{aligned} x_{Nm}(t) &= f_1(t) + x^0_{Nm}(t) \\ y_{Nm}(t) &= f_2(t) + y^0_{Nm}(t) \end{aligned} \tag{25}$$

Since $f_i(t), i = 1, 2$ are the solutions of the classical Eqs. (11) for the driven system, if we can show that $x^0_{Nm}(t)$ and $y^0_{Nm}(t)$ agree with the classical solutions of the undriven system, we will know that $x_{Nm}(t)$ and $y_{Nm}(t)$ are consistent with the classical solutions of the driven system. For mathematical simplicity, we discuss the problem in cylindrical coordinate.

In cylindrical coordinate $x = r\cos\phi$, $y = r\sin\phi$, for the undriven system the classical equation of motion is

$$\frac{d}{dt}\left(M\frac{d\vec{r}}{dt}\right) = -M\omega^2(t)\vec{r} - e\frac{d\vec{r}}{dt} \times \vec{B}_0(t) \tag{26}$$

which may be rewritten in the form

$$\begin{aligned} \frac{d}{dt}(M\dot{r}) - Mr\dot{\phi}^2 &= -M\omega^2 r - er\dot{\phi}B_0 \\ \frac{d}{dt}\left[Mr^2\left(\dot{\phi} - \frac{\varpi_c}{2}\right)\right] &= 0 \end{aligned} \tag{27}$$

The second equation has the solution $\dot{\phi} = \varpi_c/2$ or

$$\phi(t) = \int_0^t \frac{\varpi_c(\tau)}{2} d\tau \tag{28}$$

Substituting (28) into the first equation of (27), we have

$$\frac{d}{dt}(M\dot{r}) + M\Omega^2 r = 0 \tag{29}$$

which admits the solution

$$r = C\rho(t)\cos\beta(t), \quad \beta(t) = \int_0^t \frac{1}{M(\tau)\rho^2(\tau)} d\tau \tag{30}$$

Now we calculate the quantity (22) for the undriven system. Recalling that $\varphi_{nm}(\eta, \xi) = \varphi_{Nm}(R)e^{im\phi}/\sqrt{2\pi}$, it's not difficult to derive the following results by using the wave function (3)

$$(\cos\phi)^0_{Nm} = \cos\int_0^t \frac{\varpi_c(\tau)}{2} d\tau, \quad (\sin\phi)^0_{Nm} = \sin\int_0^t \frac{\varpi_c(\tau)}{2} d\tau \tag{31}$$

Hence, $(\cos\phi)^0_{Nm}$ and $(\sin\phi)^0_{Nm}$ can be written in the form $\cos\phi(t)$ and $\sin\phi(t)$, with $\phi(t)$ the classical solution (28). In another word, for the azimuthal angle, the quantity (22) and the classical solution are consistent with each other. Next let's turn to the radial part. To calculate the quantity (22) for the radial coordinate, we use the raising and lowering operators for the radial Eq. (7) (Liu *et al.*, 1997)

$$A_-(m) = R + \frac{d}{dR} + \frac{m}{R}, \quad A_+(m) = R - \frac{d}{dR} + \frac{m}{R}$$
$$B_-(m) = R + \frac{d}{dR} - \frac{m}{R}, \quad B_+(m) = R - \frac{d}{dR} - \frac{m}{R} \tag{32}$$

From these raising and lowering operators, the following relations can be proved

$$\begin{aligned} A_-(m+1)A_+(m)|Nm\rangle &= (2N+2m+4)|Nm\rangle \\ A_+(m-1)A_-(m)|Nm\rangle &= (2N+2m)|Nm\rangle \\ B_-(m-1)B_+(m)|Nm\rangle &= (2N-2m+4)|Nm\rangle \\ B_+(m+1)B_-(m)|Nm\rangle &= (2N-2m)|Nm\rangle \end{aligned} \tag{33}$$

where the Dirac note $|Nm\rangle$ for $\varphi_{Nm}(R)$ has been used. By (33), one can find that $A_-(m)|Nm\rangle$ and $|N-1, m-1\rangle$ are both the eigenfunctions of the operator $A_-(m)A_+(m-1)$ So, there exists the relation

$$A_-(m)|Nm\rangle = a^-_{Nm}|N-1, m-1\rangle \tag{34}$$

with a^-_{Nm} being a constant. Similar arguments give other relations

$$\begin{aligned} A_+(m)|Nm\rangle &= a^+_{Nm}|N+1, m+1\rangle \\ B_-(m)|Nm\rangle &= b^-_{Nm}|N-1, m+1\rangle \\ B_+(m)|Nm\rangle &= b^+_{Nm}|N+1, m-1\rangle \end{aligned} \tag{35}$$

The constant a^-_{Nm} can be found in the following way

$$\begin{aligned} |a^-_{Nm}|^2 &= \langle Nm|[A_-(m)]^+ A_-(m)|Nm\rangle \\ &= \langle Nm|A_+(m-1)A_-(m)|Nm\rangle \\ &= 2N + 2m \end{aligned} \tag{36}$$

where the relation $(\frac{d}{dR})^+ = -\frac{d}{dR} - \frac{1}{R}$ is used for the cylindrical coordinate. Other constants are also found by similar methods

$$\begin{aligned} a^+_{Nm} &= \sqrt{2N+2m+4} \\ b^-_{Nm} &= \sqrt{2N-2m} \\ b^+_{Nm} &= \sqrt{2N-2m+4} \end{aligned} \tag{37}$$

From (32), the radial coordinate is expressed as the sum of the raising and lowering operators

$$R = \frac{1}{4}[A_-(m+1) + A_+(m-1) + B_-(m-1) + B_+(m+1)] \tag{38}$$

Using (38) and the relations (34,35), we finally get

$$\begin{aligned} R_{Nm}^0(t) &= \mathrm{Re} \sum_{N'm'} \left\langle \psi_{N'm'}^0(t) | R | \psi_{Nm}^0(t) \right\rangle \\ &= \mathrm{Re} \sum_{N'm} \langle N'm | R | Nm \rangle \, \exp[i(\alpha_{Nm} - \alpha_{N'm})] \\ &= \frac{1}{4}(a_{Nm+1}^- + a_{Nm-1}^+ + b_{Nm-1}^- + b_{Nm+1}^+) \cos \int_0^t \frac{d\tau}{M(\tau)\rho^2(\tau)} \end{aligned} \tag{39a}$$

As $r = \sqrt{x^2 + y^2} = \rho\sqrt{\hbar}\sqrt{\eta^2 + \xi^2} = \rho\sqrt{\hbar}R$, hence for the coordinate r

$$r_{Nm}^0(t) = \rho(t)\sqrt{\hbar} R_{Nm}^0(t) \tag{39b}$$

In the classical limit, the quantum numbers become very large and $N\hbar$ or $m\hbar$ becomes macroscopic scale, (39) reduces to the classical solution (30).

From (6) and (28,29), we automatically get relations between the quantum and classical phases

$$\phi(t) = -\frac{\partial}{\partial m}\alpha_{Nm}^{(1)}(t), \quad \beta(t) = -\frac{\partial}{\partial N}\alpha_{Nm}^{(2)}(t) \tag{40}$$

4. CONCLUSIONS

For the time-dependent driven harmonic oscillator in a static magnetic field, it was shown that the geometrical phase induced by the driving force is the phase of the de Brolie wave associated with the particle moving according to the classical equation. Meanwhile, the exact classical solution is derived from quantum matrix elements.

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